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Binary Vectors with Prescribed Subsets of Consecutive Ones

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Recursion formulas and generating functions are derived for determining the number of binary vectors in n-space containing exactly k isolated m-tuples of consecutive ones.

1. INTRODUCTION

This paper treats a combinatorial problem on counting the number of binary vectors with prescribed subsets of consecutive ones. The research originated from an engineering problem concerned with short-circuiting of adjacent electrodes (see [1, 2]).

We consider binary vectors in n-space, that is, vectors $(x_1, ..., x_n)$ in which each component x_i is either 0 or 1. An entry $x_i = 1$ is said to be an *isolated singleton* if its neighboring components x_{i-1} and x_{i+1} are 0. If $x_1 = 1$ we only require $x_2 = 0$, and if $x_n = 1$ we only require $x_{n-1} = 0$.

A pair of consecutive ones $x_i = x_{i+1} = 1$ is said to be an *isolated pair* if $x_{i-1} = 0$ and $x_{i+2} = 0$, with the obvious modification if $x_1 = x_2 = 1$ or if $x_{n-1} = x_n = 1$.

An *isolated m-tuple* of consecutive ones $x_i = x_{i+1} = \cdots = x_{i+m-1} = 1$ is similarly defined.

We are interested in the number of binary vectors in n-space containing exactly k isolated m-tuples of consecutive ones. Recursion formulas and generating functions are derived for determining this number. The values are tabulated for $n \leqslant 25$, $0 \leqslant k \leqslant 5$, and $1 \leqslant m \leqslant 5$.

There is some overlap in this paper with the work of Guibas and Odlyzko [3] who consider a more general class of problems. Specifically, our generating function in Theorem 13 can be extracted from a general

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class of generating functions given in Eq. (1.4) of [3]. However, the treatment of the special problem considered here is more elementary and direct, and the results are presented in a way that makes them more accessible for applications.

2. Isolated Singletons

Let $S_k(n)$ denote the number of binary vectors in n-space containing exactly k isolated singletons. We begin with the case $k=0$ in which there are no isolated singletons. By direct enumeration we find

$$S_0(1)=1, \qquad S_0(2)=2, \qquad S_0(3)=4, \qquad S_0(4)=7.$$

For example, the seven vectors in 4-space with no isolated ones are

$$(0,0,0,0), \qquad (0,0,1,1), \qquad (1,1,0,0), \qquad (0,1,1,0),$$
$$(0,1,1,1), \qquad (1,1,1,0), \qquad (1,1,1,1).$$

Subsequent values of $S_0(n)$ can be determined from the following recursion.

THEOREM 1. *If $n \geqslant 3$ we have*

$$S_0(n+1)=2S_0(n)-\Delta S_0(n-1), \tag{1}$$

where Δ is the difference operator, $\Delta f(n)=f(n)-f(n-1)$.

Proof. Suppose $(x_1, ..., x_n)$ has no isolated ones, and consider the last component, x_n. If $x_n=0$, then $(x_1, ..., x_{n-1})$ has no isolated ones, and there are exactly $S_0(n-1)$ such vectors.

If $x_n=1$, we examine x_{n-1}. We cannot have $x_{n-1}=0$, otherwise $x_n=1$ would be isolated, so $x_{n-1}=1$. Now consider x_{n-2}. If $x_{n-2}=0$, then the vector $(x_1, ..., x_{n-3})$ has no isolated singletons, and there are exactly $S_0(n-3)$ such vectors. If $x_{n-2}=1$, we examine x_{n-3}. If $x_{n-3}=0$, then $(x_1, ..., x_{n-4})$ has no isolated singletons, and there are exactly $S_0(n-4)$ such vectors. We continue this process until we examine x_2. At this stage the vector has the form $(x_1, x_2, 1, ..., 1)$. If $x_2=0$, then $x_1=0$ (because $S_0(1)=1$), and if $x_2=1$, then x_1 can be either 1 or 0, so there are two vectors in this case. This argument shows that

$$S_0(n)=S_0(n-1)+S_0(n-3)+S_0(n-4)+\cdots+S_0(1)+2. \tag{2}$$

The term $S_0(n-2)$ does not appear on the right because the case in which $x_n=1$ and $x_{n-1}=0$ was excluded. Now replace n by $n+1$ in Eq. (2) to obtain

$$S_0(n+1)=S_0(n)+S_0(n-2)+S_0(n-3)+\cdots+S_0(1)+2. \tag{3}$$

Subtracting Eq. (2) from (3) we obtain the recursion formula in Theorem 1. The values of $S_0(n)$ for $n \leqslant 25$ are listed in Table I.

We turn next to $S_1(n)$, the number of binary vectors with exactly one isolated singleton. Direct enumeration shows that

$$S_1(1) = 1, \qquad S_1(2) = 2, \qquad S_1(3) = 3, \qquad S_1(4) = 6.$$

The remaining values can be determined by the following recursion.

THEOREM 2. *For $n \geqslant 3$ we have*

$$S_1(n+1) = 2S_1(n) - \Delta S_1(n-1) + \Delta S_0(n-1). \tag{4}$$

Proof. By an argument similar to that given for Theorem 1 we find

$$S_1(n) = S_1(n-1) + S_0(n-2) + S_1(n-3) + S_1(n-4) + \cdots + S_1(1).$$

Subtract this equation from the same relation with n replaced by $n+1$ to obtain Eq. (4). Table I also lists values of $S_1(n)$ for $n \leqslant 25$.

Next, we consider $S_2(n)$, the number of binary vectors with exactly two isolated singletons. Direct enumeration shows that

$$S_2(1) = S_2(2) = 0, \qquad S_2(3) = 1, \qquad S_2(4) = 3, \qquad S_2(5) = 6.$$

Subsequent values can be determined from the following recursion.

THEOREM 3. *For $n \geqslant 3$ we have*

$$S_2(n+1) = 2S_2(n) - \Delta S_2(n-1) + \Delta S_1(n-1). \tag{5}$$

The proof is like that for Theorem 2. In fact, exactly the same argument gives the following general theorem for $S_k(n)$ when $2 \leqslant k \leqslant [(n+1)/2]$.

THEOREM 4. *For $k \geqslant 2$ and $n \geqslant 3$ we have*

$$S_k(n+1) = 2S_k(n) - \Delta S_k(n-1) + \Delta S_{k-1}(n-1), \tag{6}$$

with initial values

$$S_k(1) = S_k(2) = \cdots = S_k(2k-2) = 0, \qquad S_k(2k-1) = 1, \qquad S_k(2k) = k+1.$$

Values of $S_2(n)$, $S_3(n)$, $S_4(n)$, $S_5(n)$ are given in Table I.

TABLE I

$S_k(n)$ = Number of Binary Vectors in n-Space with Exactly k Isolated Singletons

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$S_0(n) =$	1	2	4	7	12	21	37	65	114	200	351	616	1081	1897	3329
$S_1(n) =$	1	2	3	6	13	26	50	96	184	350	661	1242	2324	4332	8047
$S_2(n) =$	0	0	1	3	6	13	30	66	139	288	591	1199	2406	4785	9445
$S_3(n) =$	0	0	0	0	1	4	10	24	59	140	318	704	1533	3288	6954
$S_4(n) =$	0	0	0	0	0	0	1	5	15	40	105	266	645	1515	3480
$S_5(n) =$	0	0	0	0	0	0	0	0	1	6	21	62	174	468	1203

$n =$	16	17	18	19	20	21	22	23	24	25
$S_0(n) =$	5842	10252	17991	31572	55405	97229	170625	299426	525456	922111
$S_1(n) =$	14902	27521	50700	93191	170942	312974	572030	1043852	1902044	3461067
$S_2(n) =$	18519	36093	69967	134979	259263	496005	945477	1796244	3402072	6425199
$S_3(n) =$	14532	30058	61612	125272	252864	507080	1010852	2004230	3954160	7765709
$S_4(n) =$	7845	17391	37995	81970	174890	369465	773602	1606845	3313325	6786785
$S_5(n) =$	2982	7194	16974	39282	89388	200443	443700	970920	2102720	4511463

3. Isolated Pairs

Let $P_k(n)$ denote the number of binary vectors in n-space containing exactly k isolated pairs of consecutive ones. We begin with the case $k=0$. By direct enumeration we find

$$P_0(1)=2, \qquad P_0(2)=3, \qquad P_0(3)=6, \qquad P_0(4)=11.$$

Subsequent values can be determined from the following recursion formula.

Theorem 5. *If $n \geqslant 3$ we have*

$$P_0(n+1)=2P_0(n)-\Delta P_0(n-2). \tag{7}$$

Proof. Suppose the binary vector $(x_1, ..., x_n)$ has no isolated pair, and consider the last entry, x_n.

If $x_n=0$, then $(x_1, ..., x_{n-1})$ has no isolated pair, and there are exactly $P_0(n-1)$ such vectors.

If $x_n=1$, we examine x_{n-1}. If $x_{n-1}=0$, then $(x_1, ..., x_{n-2})$ has no isolated pair, and there are exactly $P_0(n-2)$ such vectors.

If $x_n=x_{n-1}=1$, the vector is $(x_1, ..., x_{n-3}, x_{n-2}, 1, 1)$ and we examine x_{n-2}. We cannot have $x_{n-2}=0$, otherwise the last two entries would be an isolated pair. Therefore, $x_{n-2}=1$ and the vector is $(x_1, ..., x_{n-3}, 1, 1, 1)$. Now examine x_{n-3}. If $x_{n-3}=0$, then $(x_1, ..., x_{n-4})$ has no isolated pair, and there are exactly $P_0(n-4)$ such vectors. If $x_{n-3}=1$, we examine x_{n-4}. Repeat the process until we arrive at the vector $(x_1, 1, 1, ..., 1)$. There are $2=P_0(1)$ choices for x_1, each of which results in no isolated pair, so we have

$$P_0(n)=P_0(n-1)+P_0(n-2)+P_0(n-4)+ \cdots +P_0(1).$$

Subtract this from the same relation in which n is replaced by $n+1$ to obtain Eq. (7). The values of $P_0(n)$ for $n \leqslant 25$ are listed in Table II.

Next we consider $P_1(n)$, the number of binary vectors with exactly one isolated pair of consecutive ones. By direct enumeration we find that

$$P_1(1)=0, \qquad P_1(2)=1, \qquad P_1(3)=2, \qquad P_1(4)=5.$$

Subsequent values can be determined from the following recursion.

Theorem 6. *For $n \geqslant 3$ we have*

$$P_1(n+1)=2P_1(n)-\Delta P_1(n-2)+\Delta P_0(n-2). \tag{8}$$

Proof. By an argument similar to that given for Theorem 5 we find

$$P_1(n)=P_1(n-1)+P_1(n-2)+P_0(n-3)+P_1(n-4)+ \cdots +P_1(1).$$

TABLE II

$P_k(n)$ = Number of Binary Vectors in n-Space with Exactly k Isolated Pairs of Consecutive Ones

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$P_0(n) =$	2	3	6	11	21	39	73	136	254	474	885	1652	3084	5757	10747
$P_1(n) =$	0	1	2	5	10	22	46	97	200	410	832	1679	3368	6725	13370
$P_2(n) =$	0	0	0	0	1	3	9	22	54	126	290	651	1440	3138	6762
$P_3(n) =$	0	0	0	0	0	0	0	1	4	14	40	109	280	698	1688
$P_4(n) =$	0	0	0	0	0	0	0	0	0	0	1	5	20	65	195
$P_5(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	6

$n =$	16	17	18	19	20	21	22	23	24	25
$P_0(n) =$	20062	37451	69912	130509	243629	454797	848997	1584874	2958580	5522960
$P_1(n) =$	26483	52282	102909	202020	395630	773094	1507674	2934858	5703420	11066460
$P_2(n) =$	14424	30507	64035	133521	276758	570615	1170855	2392083	4867773	9869886
$P_3(n) =$	3994	9268	21170	47696	106201	234028	511016	1106764	2379558	5082368
$P_4(n) =$	546	1465	3790	9535	23425	56427	133635	311885	718595	1636970
$P_5(n) =$	27	98	321	972	2792	7686	20466	53002	134112	332652

Subtract this equation from the same relation with n replaced by $n+1$ to obtain Eq. (8).

Next we treat $P_2(n)$, the number of binary vectors with exactly two isolated pairs of consecutive ones. By direct enumeration we find

$$P_2(n)=0 \quad \text{for} \quad n=1, 2, 3, 4, \qquad P_2(5)=1, \qquad P_2(6)=3.$$

Subsequent values can be determined from the following theorem whose proof is similar to that of Theorem 6.

THEOREM 7. *For $n \geqslant 3$ we have*

$$P_2(n+1)=2P_2(n)-\Delta P_2(n-2)+\Delta P_1(n-2). \tag{9}$$

More generally, for $k \geqslant 2$ the recursion formula for $P_k(n)$ has the following form.

THEOREM 8. *For $k \geqslant 2$ and $n \geqslant 3$ we have*

$$P_k(n+1)=2P_k(n)-\Delta P_k(n-2)+\Delta P_{k-1}(n-2), \tag{10}$$

with initial values

$$P_k(n)=0 \quad \text{for} \quad n=1, 2, ..., 3k-2 \quad \text{and} \quad P_k(3k-1)=1.$$

The values of $P_k(n)$ for $0 \leqslant k \leqslant 5$ and $1 \leqslant n \leqslant 25$ are listed in Table II.

4. ISOLATED m-TUPLES

The same type of analysis can be applied to triples, quadruples, and, in general, m-tuples. Let $A_k^{(m)}(n)$ (Tables III–V) denote the number of binary vectors in n-space with exactly k isolated m-tuples of consecutive ones, where $m>1$. Consider first the case $k=0$.

If $n<m$, no n-vector can contain an m-tuple of ones, so $A_0^{(m)}(n)$ is the total number of binary n-vectors,

$$A_0^{(m)}(n)=2^n \quad \text{if} \quad n=1, 2, ..., m-1. \tag{11}$$

If $n=m$, there is only one n-vector with all entries equal to 1, so

$$A_0^{(m)}(m)=2^m-1. \tag{12}$$

If $n=m+1$, there are exactly two n-vectors $(0, 1, ..., 1)$ and $(1, ..., 1, 0)$ containing an isolated m-tuple of consecutive ones, so in this case

$$A_0^{(m)}(m+1)=2^{m+1}-2. \tag{13}$$

TABLE III

$T_k(n)$ = Number of Binary Vectors in n-Space with Exactly k Isolated Triples of Consecutive Ones

$n=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$T_0(n)=$	2	4	7	14	27	52	101	195	377	729	1409	2724	5266	10180	19680
$T_1(n)=$	0	0	1	2	5	12	26	58	126	270	575	1212	2538	5284	10943
$T_2(n)=$	0	0	0	0	0	0	1	3	9	25	63	156	374	876	2019
$T_3(n)=$	0	0	0	0	0	0	0	0	0	0	1	4	14	44	125
$T_4(n)=$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$T_5(n)=$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$n=$	16	17	18	19	20	21	22	23	24	25
$T_0(n)=$	38045	73548	142182	274864	531363	1027223	1985812	3838942	7421385	14346910
$T_1(n)=$	22564	46344	94856	193553	393850	799423	1618968	3271921	6600044	13290375
$T_2(n)=$	4582	10272	22788	50092	109242	236574	509144	1089681	2320509	4919259
$T_3(n)=$	340	888	2248	5558	13464	32070	75296	174586	400416	909558
$T_4(n)=$	5	20	70	220	651	1835	4980	13120	33715	84852
$T_5(n)=$	0	0	0	1	6	27	104	357	1140	3443

TABLE IV

$A_k^{(4)}(n)$ = Number of Binary Vectors in n-Space with Exactly k Isolated 4-Tuples of Consecutive Ones

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$A_0^{(4)}(n) =$	2	4	8	15	30	59	116	228	449	883	1737	3417	6722	13223	26012
$A_1^{(4)}(n) =$	0	0	0	1	2	5	12	28	62	138	302	654	1404	2995	6348
$A_2^{(4)}(n) =$	0	0	0	0	0	0	0	0	1	3	9	25	66	165	404
$A_3^{(4)}(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4
$A_4^{(4)}(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$A_5^{(4)}(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$n =$	16	17	18	19	20	21	22	23	24	25
$A_0^{(4)}(n) =$	51170	100660	198015	389529	766269	1507380	2965270	5833185	11474856	22572972
$A_1^{(4)}(n) =$	13386	28100	58755	122420	254276	526672	1088120	2242940	4613729	9472342
$A_2^{(4)}(n) =$	966	2268	5245	11982	27078	60632	134676	297030	650988	1418736
$A_3^{(4)}(n) =$	14	44	129	356	948	2448	6168	15228	36966	88436
$A_4^{(4)}(n) =$	0	0	0	1	5	20	70	225	676	1940
$A_5^{(4)}(n) =$	0	0	0	0	0	0	0	0	1	6

TABLE V

$A_k^{(5)}(n)$ = Number of Binary Vectors in n-Space with Exactly k Isolated 5-Tuples of Consecutive Ones

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$A_0^{(5)}(n) =$	2	4	8	16	31	62	123	244	484	960	1905	3779	7497	14873	29506
$A_1^{(5)}(n) =$	0	0	0	0	1	2	5	12	28	64	142	314	686	1486	3196
$A_2^{(5)}(n) =$	0	0	0	0	0	0	0	0	0	0	1	3	9	25	66
$A_3^{(5)}(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$A_4^{(5)}(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$A_5^{(5)}(n) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$n =$	16	17	18	19	20	21	22	23	24	25
$A_0^{(5)}(n) =$	58536	116127	230380	457042	906708	1798783	3568536	7079481	14044709	27862756
$A_1^{(5)}(n) =$	6832	14531	30764	64874	136324	285571	596536	1242964	2583948	5360448
$A_2^{(5)}(n) =$	168	413	996	2358	5500	12669	28872	65198	146046	324840
$A_3^{(5)}(n) =$	0	1	4	14	44	129	360	964	2508	6368
$A_4^{(5)}(n) =$	0	0	0	0	0	0	0	1	5	20
$A_5^{(5)}(n) =$	0	0	0	0	0	0	0	0	0	0

And if $n=m+2$, there are exactly five n-vectors containing isolated m-tuples of consecutive ones, namely

$$(x, 0, 0), \qquad (x, 0, 1), \qquad (0, x, 0), \qquad (1, 0, x), \qquad (0, 0, x),$$

where x denotes the m-tuple of ones. Hence

$$A_0^{(m)}(m+2)=2^{m+2}-5. \tag{14}$$

For subsequent values of n we use the following theorem whose proof is like that of Theorem 5.

THEOREM 9. *For $n \geqslant m+2$ we have*

$$A_0^{(m)}(n+1)=2A_0^{(m)}(n)-\Delta A_0^{(m)}(n-m). \tag{15}$$

This formula, together with the initial values in (11) through (14), completely determines the numbers $A_0^{(m)}(n)$ for all n.

Next, we consider the case $k=1$. The analysis used to determine the initial values in (11) through (14) shows that we have (for $m>1$)

$$A_1^{(m)}(n)=0 \qquad \text{if} \quad n=1, 2, ..., m-1, \tag{16}$$

and

$$A_1^{(m)}(m)=1, \qquad A_1^{(m)}(m+1)=2, \qquad \text{and} \qquad A_1^{(m)}(m+2)=5. \tag{17}$$

For subsequent values of n we use the following theorem whose proof is like that of Theorem 6.

THEOREM 10. *For $n \geqslant m+2$ we have*

$$A_1^{(m)}(n+1)=2A_1^{(m)}(n)-\Delta A_1^{(m)}(n-m)+\Delta A_0^{(m)}(n-m). \tag{18}$$

The corresponding recursion formula for $A_k^{(m)}(n)$ for $k \geqslant 2$ is given in the next theorem.

THEOREM 11. *For $k \geqslant 2$, $m \geqslant 1$ and $n \geqslant km+k-1$ we have*

$$A_k^{(m)}(n+1)=2A_k^{(m)}(n)-\Delta A_k^{(m)}(n-m)+\Delta A_{k-1}^{(m)}(n-m), \tag{19}$$

with initial values

$$A_k^{(m)}(n)=0 \qquad \text{for} \quad n=1, 2, ..., km+k-2, \tag{20}$$

and

$$A_k^{(m)}(km+k-1)=1. \tag{21}$$

5. Generating Functions

This section determines generating functions for the number $A_k^{(m)}(n)$. We begin with isolated singletons, and we write $S_k(n)$ for $A_k^{(1)}(n)$.

THEOREM 12. *Let* $F_k(n)=\sum_{n=1}^{\infty} S_k(n)\,x^n$, *and let* $Q(x)=1-2x+x^2-x^3$. *Then we have*

$$F_0(x)=\frac{x+x^3}{Q(x)} \tag{22}$$

and, for $k\geqslant 1$,

$$F_k(x)=\frac{x^{2k-1}(1-x)^{k+1}}{Q(x)^{k+1}}. \tag{23}$$

Proof. When $k=0$ we have

$$F_0(x)=\sum_{n=1}^{\infty} S_0(n)\,x^n=x+2x^2+4x^3+\sum_{n=3}^{\infty} S_0(n+1)\,x^{n+1}.$$

Using the recursion in Theorem 1 we find

$$\begin{aligned}F_0(x)&=x+2x^2+4x^3+\sum_{n=3}^{\infty}\{2S_0(n)-S_0(n-1)+S_0(n-2)\}\,x^{n+1}\\&=x+2x^2+4x^3+2x\{F_0(x)-x-2x^2\}-x^2\{F_0(x)-x\}+x^3F_0(x)\\&=x+x^3+(2x-x^2+x^3)\,F_0(x).\end{aligned}$$

Solving this last equation for $F_0(x)$ we obtain (22). Next we take $k=1$ so that

$$F_1(x)=\sum_{n=1}^{\infty} S_1(n)\,x^n=x+2x^2+3x^3+\sum_{n=3}^{\infty} S_1(n+1)\,x^{n+1}.$$

Using the recursion in Theorem 2 together with that in (1) we obtain

$$\begin{aligned}F_1(x)&=x+2x^2+3x^3+\sum_{n=3}^{\infty}\{2S_1(n)-S_1(n-1)\\&\qquad+S_1(n-2)+2S_0(n)-S_0(n+1)\}\,x^{n+1}\\&=x+2x^2+3x^3+2x\{F_1(x)-x-2x^2\}\\&\qquad-x^2\{F_1(x)-x\}+x^3F_1(x)+2xF_0(x)-\{F_0(x)-x\}.\end{aligned}$$

Hence

$$F_1(x)(1-2x+x^2-x^3)=(2x-1)\,F_0(x)+2x.$$

Using (22) we obtain

$$F_1(x)=\frac{(2x-1)(x+x^3)+2xQ(x)}{Q(x)^2}=\frac{x^3-2x^2+x}{Q(x)^2}=\frac{x(1-x)^2}{Q(x)^2}.$$

This proves (23) when $k=1$. The result for general $k\geqslant 1$ will be proved by induction on k.

We recall that $S_k(1)=S_k(2)=0$ for $k\geqslant 2$. Using the recursion in Theorem 4 we find

$$\begin{aligned}F_k(x)&=\sum_{n=2}^{\infty} S_k(n+1)\,x^{n+1}\\&=\sum_{n=2}^{\infty}\{2S_k(n)-S_k(n-1)+S_k(n-2)\\&\qquad+S_{k-1}(n-1)-S_{k-1}(n-2)\}\,x^{n+1}\\&=(2x-x^2+x^3)\,F_k(x)+(x^2-x^3)\,F_{k-1}(x),\end{aligned}$$

so $Q(x)\,F_k(x)=(x^2-x^3)\,F_{k-1}(x)$ or $F_k(x)=x^2(1-x)\,F_{k-1}(x)/Q(x)$. This, together with the formula for $F_1(x)$, proves (23) by induction for all $k\geqslant 1$.

THEOREM 13. *If $m\geqslant 2$ and $k\geqslant 0$, let $F_k^{(m)}(x)=\sum_{n=1}^{\infty} A_k^{(m)}(n)\,x^n$, and let $Q(x)=1-2x+x^{m+1}-x^{m+2}$. Then we have*

$$F_0^{(m)}(x)=\frac{x(2-x^{m-1}+x^{m+1})}{Q_m(x)} \tag{24}$$

and, for $k\geqslant 1$,

$$F_k^{(m)}(x)=\frac{x^{km+k-1}(1-x)^{k+1}}{Q_m(x)^{k+1}}. \tag{25}$$

Proof. Using the initial values in (11) through (14) we see that

$$\begin{aligned}F_0^{(m)}(x)&=2x+4x^2+\cdots+2^{m-1}x^{m-1}+(2^m-1)\,x^m+(2^{m+1}-2)\,x^{m+1}\\&\quad+(2^{m+2}-5)\,x^{m+2}+\sum_{n=m+3}^{\infty} A_0^{(m)}(n)\,x^n\\&=\sum_{n=1}^{m+2}(2x)^n-x^m(1+2x+5x^2)+\sum_{n=m+2}^{\infty} A_0^{(m)}(n+1)\,x^{n+1}.\end{aligned}$$

To evaluate the last series we use the recursion formula in (15) together with the initial values in (11) through (14) to obtain

$$\begin{aligned}
&\sum_{n=m+2}^{\infty} A_0^{(m)}(n+1)\,x^{n+1}\\
&\quad=\sum_{n=m+2}^{\infty}\{2A_0^{(m)}(n)-A_0^{(m)}(n-m)+A_0^{(m)}(n-1-m)\}\,x^{n+1}\\
&\quad=2x\sum_{n=m+2}^{\infty} A_0^{(m)}(n)\,x^n-x^{m+1}\sum_{n=m+2}^{\infty} A_0^{(m)}(n-m)\,x^{n-m}\\
&\qquad+x^{m+2}\sum_{n=m+2}^{\infty} A_0^{(m)}(n-1-m)\,x^{n-1-m}\\
&\quad=2x\left\{F_0^{(m)}(x)-\sum_{n=1}^{m+1}(2x)^n+x^m(1+2x)\right\}\\
&\qquad-x^{m+1}\{F_0^{(m)}(x)-2x\}+x^{m+2}F_0^{(m)}(x).
\end{aligned}$$

Using this in the above equation for $F_0^{(m)}(x)$ we find

$$\begin{aligned}
F_0^{(m)}(x)\,Q_m(x)&=\sum_{n=1}^{m+2}(2x)^n-x^m(1+2x+5x^2)\\
&\quad-\sum_{n=2}^{m+2}(2x)^n+x^m(2x+4x^2)+2x^{m+2}\\
&=2x-x^m(1+x^2)+2x^{m+2}=x(2-x^{m-1}+x^{m+1}),
\end{aligned}$$

which implies (24).

Next we take $k=1$. Using the initial values in (16) and (17) we find

$$\begin{aligned}
F_1^{(m)}(x)&=x^m+2x^{m+1}+5x^{m+2}+\sum_{n=m+2}^{\infty} A_1^{(m)}(n+1)\,x^{n+1}\\
&=x^m+2x^{m+1}+5x^{m+2}\\
&\quad+\sum_{n=m+2}^{\infty}\{2A_1^{(m)}(n)-A_1^{(m)}(n-m)+A_1^{(m)}(n-1-m)\\
&\quad+A_0^{(m)}(n-m)-A_0^{(m)}(n-1-m)\}\,x^{n+1}\\
&=x^m+2x^{m+1}+5x^{m+2}+2x\{F_1^{(m)}(x)-x^m-2x^{m+1}\}-x^{m+1}F_1^{(m)}(x)\\
&\quad+x^{m+2}F_1^{(m)}(x)+x^{m+1}\{F_0^{(m)}(x)-2x\}-x^{m+2}F_0^{(m)}(x).
\end{aligned}$$

Therefore,

$$\begin{aligned} F_1^{(m)}(x)\, Q_m(x) &= x^m - x^{m+2} + F_0^{(m)}(x)(x^{m+1} - x^{m+2}) \\ &= x^m(1 - x^2) + x^{m+1}(1 - x)(2x - x^m + x^{m+2})/Q_m(x) \\ &= \frac{x^m(1-x)\{(1+x)\, Q_m(x) + 2x^2 - x^{m+1} + x^{m+3}\}}{Q_m(x)} \\ &= \frac{x^m(1-x)^2}{Q_m(x)} \end{aligned}$$

so

$$F_1^{(m)}(x) = \frac{x^m(1-x)^2}{Q_m(x)^2}. \tag{26}$$

If $k \geqslant 2$, a similar argument shows that

$$F_k^{(m)}(x) = \frac{x^{m+1}(1-x)}{Q_m(x)} F_{k-1}^{(m)}(x),$$

and when this is combined with (26) we obtain a proof of (25) by induction on k.

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